11 February 2011, 14:00 – 17:00, X 5118.-156

Rijksuniversiteit Groningen
Statistiek

Hertentamen

RULES FOR THE EXAM:

- The use of a normal, non-graphical calculator is permitted.
- This is a CLOSED-BOOK exam.
- At the end of the exam you can find a normal table and a chi-squared table.

1. **Point estimation (1).** Let $X_1, \ldots, X_n$ be a sample of independent, identically distributed random variables, with density

$$f_\theta(x) = \begin{cases} \frac{2x}{\theta^2} & 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Determine the maximum likelihood estimator (MLE) of θ. [5 Marks]

   (b) Let $\hat{\theta}_n$ be the estimator of θ you derived in (a),

      i. Determine whether $\hat{\theta}_n$ is unbiased. [5 Marks]

      ii. Determine whether $\hat{\theta}_n$ is consistent. [5 Marks]

      iii. Determine whether $\hat{\theta}_n$ is sufficient. [5 Marks]

   (c) Why doesn't the Cramer-Rao lower bound apply to unbiased estimates of θ for this distribution? [5 Marks]

2. **Point estimation (2).** Let $X_1, \ldots, X_n$ be a sample of independent, identically distributed Exp(θ) random variables, with density

$$f_\theta(x) = \frac{e^{-x/\theta}}{\theta}, \quad x > 0.$$

   (a) Determine the Cramer-Rao lower bound for the variance of an unbiased estimate of θ. [5 Marks]

   (b) Determine the MLE $\hat{\theta}$ of θ. [5 Marks]

   (c) Check that $\hat{\theta}$ is unbiased an whether it attains the Cramer-Rao lower bound. [5 Marks]

   (d) In fact, we are told that $n = 3$ and that the data are given as

$$x_1 = 1, \quad x_2 = 2.5, \quad x_3 = 5.5.$$

      i. Determine an approximate 95% confidence interval based on the asymptotic normality of $\hat{\theta}$. [5 Marks]

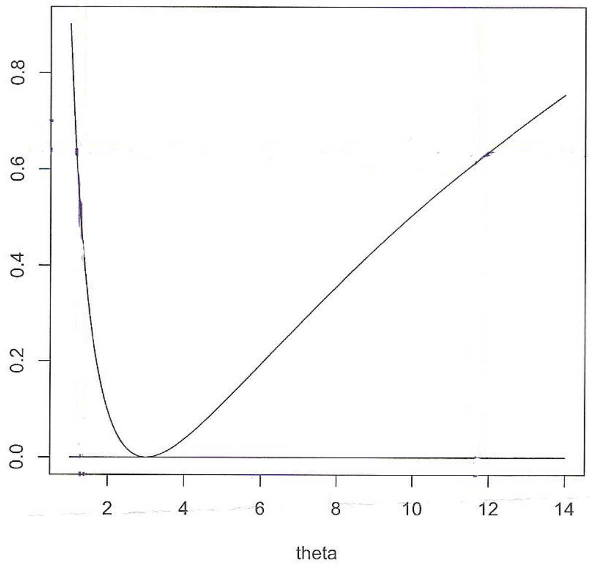

Figure 1: This is the function $g(\theta) = \frac{3}{\theta} + \log(\theta) - \log 3 - 1$.

ii. Determine an approximate 95% confidence interval based on the asymptotic distribution of the deviance $D(\theta)$. (The solution does not have to be exact and you can use Figure 1, but pay attention to the definition of be g!) [5 Marks]

3. **Hypothesis testing.**

   (a) Let X_1 and X_2 constitute a random sample from a normal population with $\sigma^2 = 1$. If the null hypothesis $\mu = \mu_0$ is to be rejected in favour of the alternative hypothesis $\mu = \mu_1 > \mu_0$ when $\bar{x} > \mu_0 + 1$, what is the size of the critical region? (question 12.7 from book) [10 Marks]

   (b) A random sample of size n from an exponential population is used to test the null hypothesis $\theta = \theta_0$ against the alternative hypothesis $\theta = \theta_1 < \theta_0$. Use the Neyman-Pearson lemma to find the shape of the most powerful critical region of size α. (Work out as far as you can. Use the fact that the sum of n independent Exp(θ) random variables is Gamma(n, θ) distributed. A α resp. $(1-\alpha)$ quantile of such Gamma distribution can be written as $\Gamma_{\alpha;n,\theta}$ resp. $\Gamma_{1-\alpha;n,\theta}$). [10 Marks]

Next page contains statistical tables which may be used in the calculations.

$\nu \setminus \alpha$	0.995	0.99	0.975	0.95	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	11.070	12.833	15.086	16.750
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188

Table 1: Values of $\chi^2_{\alpha,\nu}$ as found in the book: the entries in the table correspond to values of x, such that $P(\chi^2_\nu > x) = \alpha$, where χ^2_ν correspond to a chi-squared distributed variable with ν degrees of freedom.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.000	0.004	0.008	0.012	0.016	0.020	0.024	0.028	0.032	0.036
0.1	0.040	0.044	0.048	0.052	0.056	0.060	0.064	0.067	0.071	0.075
0.2	0.079	0.083	0.087	0.091	0.095	0.099	0.103	0.106	0.110	0.114
0.3	0.118	0.122	0.126	0.129	0.133	0.137	0.141	0.144	0.148	0.152
0.4	0.155	0.159	0.163	0.166	0.170	0.174	0.177	0.181	0.184	0.188
0.5	0.191	0.195	0.198	0.202	0.205	0.209	0.212	0.216	0.219	0.222
0.6	0.226	0.229	0.232	0.236	0.239	0.242	0.245	0.249	0.252	0.255
0.7	0.258	0.261	0.264	0.267	0.270	0.273	0.276	0.279	0.282	0.285
0.8	0.288	0.291	0.294	0.297	0.300	0.302	0.305	0.308	0.311	0.313
0.9	0.316	0.319	0.321	0.324	0.326	0.329	0.331	0.334	0.336	0.339
1.0	0.341	0.344	0.346	0.348	0.351	0.353	0.355	0.358	0.360	0.362
1.1	0.364	0.367	0.369	0.371	0.373	0.375	0.377	0.379	0.381	0.383
1.2	0.385	0.387	0.389	0.391	0.393	0.394	0.396	0.398	0.400	0.401
1.3	0.403	0.405	0.407	0.408	0.410	0.411	0.413	0.415	0.416	0.418
1.4	0.419	0.421	0.422	0.424	0.425	0.426	0.428	0.429	0.431	0.432
1.5	0.433	0.434	0.436	0.437	0.438	0.439	0.441	0.442	0.443	0.444
1.6	0.445	0.446	0.447	0.448	0.449	0.451	0.452	0.453	0.454	0.454
1.7	0.455	0.456	0.457	0.458	0.459	0.460	0.461	0.462	0.462	0.463
1.8	0.464	0.465	0.466	0.466	0.467	0.468	0.469	0.469	0.470	0.471
1.9	0.471	0.472	0.473	0.473	0.474	0.474	0.475	0.476	0.476	0.477
2.0	0.477	0.478	0.478	0.479	0.479	0.480	0.480	0.481	0.481	0.482
2.1	0.482	0.483	0.483	0.483	0.484	0.484	0.485	0.485	0.485	0.486
2.2	0.486	0.486	0.487	0.487	0.487	0.488	0.488	0.488	0.489	0.489
2.3	0.489	0.490	0.490	0.490	0.490	0.491	0.491	0.491	0.491	0.492
2.4	0.492	0.492	0.492	0.492	0.493	0.493	0.493	0.493	0.493	0.494
2.5	0.494	0.494	0.494	0.494	0.494	0.495	0.495	0.495	0.495	0.495
2.6	0.495	0.495	0.496	0.496	0.496	0.496	0.496	0.496	0.496	0.496
2.7	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497
2.8	0.497	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498
2.9	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.499	0.499	0.499
3.0	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499

Table 2: Standard Normal Distribution as found in the book. This means that values in the table correspond to probabilities $P(0 < Z \leq z)$, where Z is a standard normal distributed variable.

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